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# Algebraic quantisation with indefinite metric

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**Abstract.** A real symplectic flow may be viewed as being pseudo-unitary, provided there exists a symplectic complex structure $J$ which commutes with it. This paper determines the class of dynamical systems for which $J$ exists. Such systems may be quantised algebraically but the state space may have indefinite metric. This procedure admits a larger class of classical systems than does Segal's algebraic quantisation but there are problems in interpreting the indefinite metric and these are listed and discussed.

## 1. Introduction

The indefinite metric became familiar in quantum field theory after it had been demonstrated by Gupta and independently by Bleuler that a covariant formulation of quantum electrodynamics necessitated the introduction of a negative metric for the unphysical longitudinal and time-like photons (e.g. Mandl 1959). There is now a general result, due to Strocchi (1978), that any covariant locally gauge invariant field must be represented on an indefinite inner product space. The indefinite metric also arises when a covariant wave equation, with minimal coupling to an external field, becomes unstable. In such circumstances, it is unavoidable that formal mode space should become indefinite (Krajcik and Nieto 1976, Gupta 1978, Barua and Gupta 1978, Broadbridge 1981). A third circumstance which leads to the indefinite metric is the quantisation of a non-local field. Non-local fields are still considered as models for strong interactions, either to confine the sub-hadronic particles (e.g. d'Emilio and Mintchev 1979) or to take account of their spatial extension (Broadbridge 1981 and references therein).

The emergence of the indefinite metric can already be accounted for at the foundation level of quantum mechanics, in the theory of simple quadratic Hamiltonians. Just as the free Klein–Gordon field can be analysed as a collection of independent harmonic oscillators, fields with non-local action are made up of indecomposable finite linear subsystems of the most general type, as shown by Pais and Uhlenbeck (1950). Such a system has a general quadratic Hamiltonian

$$H=\tfrac{1}{2}z^{\mathrm{T}}\hat{H}z, \tag{1.1}$$

with $z^{\mathrm{T}}=(q,p)\in R^{2N}$ and $\hat{H}=\hat{H}^{\mathrm{T}}$, a real symmetric $2N\times 2N$ matrix. The Hamilton

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equations may then be solved to yield

$$z(t) = \exp(-G\hat{H}t)z(0), \tag{1.2}$$

with

$$G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The single-parameter group $C(t) = \exp(-G\hat{H}t)$ is a subgroup of the real symplectic group $\mathrm{Sp}(2N, R)$, which preserves the symplectic form $B(z, z') = (z, Gz')$. The generator $-G\hat{H}$ belongs to the Lie algebra $\mathrm{sp}(2N, R)$. From (1.2), $z(t)$ may be expressed as an exponential polynomial

$$z_\mu(t) = \sum_k \sum_{l=0}^{N_k-1} b_{\mu,k,l} t^l \, e^{is_k t} + \text{complex conjugate}. \tag{1.3}$$

In (1.3) the $k$-summation accounts for all elementary divisors $(s - s_k)^{N_k}$ of $-iG\hat{H} - sI$. Since $z(t)$ is determined by its initial value, there are $2N$ independent parameters among the coefficients $b_{\mu,k,l}$. To quantise (1.3) in the Heisenberg scheme, $z_j(t)$ is replaced by a time-dependent operator $Z_j(t)$, while the coefficients $b, b^*$ become constant operators $B, B^\dagger$. This straightforward substitution follows for the quadratic Hamiltonians, since in this case the Heisenberg equations have the same form as the classical Hamilton equations. For a collection of harmonic oscillators, $s_k$ is real non-zero and $N_k = 1$. In this case, the canonical commutation relations (CCR)

$$[Z_\mu(t), Z_\nu(t)] = -iG_{\mu\nu} \tag{1.4}$$

are equivalent to the boson commutation relations (BCR)

$$[B_j, B_k^\dagger] = \delta_{jk} I \qquad \text{and} \qquad [B_j, B_k] = 0. \tag{1.5}$$

If we then assume the existence of a cyclic normalised vacuum state $\psi_0$ such that $B_j \psi_0 = 0$, formal single-particle space has its familiar positive definite metric $\langle B_j^\dagger \psi_0, B_k^\dagger \psi_0 \rangle = \delta_{jk}$. However, when some of the frequencies $s_k$ are complex or some of the multiplicities $N_k$ exceed one, the dynamical system is unstable and cannot be transformed linearly to a collection of independent harmonic oscillators. In all such cases, the CCR are incompatible with the BCR. If we choose to retain the BCR, then we obtain acausal commutation relations among the canonical operators. For example, such relations result from the imaginary mass Klein–Gordon system and this has been used to model a tachyon field (Sudarshan and Dhar 1968, Arons and Sudarshan 1968). In fact, the classical imaginary mass Klein–Gordon system, based on the equation

$$0 = (\partial_t^2 - \nabla^2 - m^2)(x),$$

may be analysed as a collection of independent harmonic oscillators together with a set of repulsive (pure imaginary frequency) systems. For a charged repulsive oscillator,

$$H = P^\dagger P - Q^\dagger Q,$$

with

$$Q, Q^\dagger = 2^{-1/2}(Q_1 \mp iQ_2) \qquad \text{and} \qquad P, P^\dagger = 2^{-1/2}(P_1 \pm iP_2),$$

we obtain, after assuming (1.5), (Broadbridge 1982)

$$[Q_1, Q_2] = -2i \cosh 2t. \tag{1.6}$$

Commutators of the type (1.6) are the finite-dimensional prototypes of acausal non-vanishing commutators among field operators $\varphi(x)$, $\varphi(y)$ with $(x-y)$ space-like. Schroer (1971) quantised this system using the other approach, namely to assume the CCR among $Q_j$ and $P_j$ and to deduce the commutation relations among mode operators $B_j$ and $B_j^\dagger$. There is now a general result that for any unstable linear system, this approach must lead to an indefinite metric for the single-mode space spanned by $B_j^\dagger\psi_0$ (Broadbridge 1981). Hence, we see why the indefinite metric arises in the external field problem and also in non-local field theory.

Of course, a quadratic quantum mechanical Hamiltonian can always be expressed as a quadratic boson Hamiltonian, for example by defining annihilation operators $b_j = 2^{-1/2}(Q_j + \mathrm{i}P_j)$ and creation operators $b_j^\dagger$. However, these boson construction operators may not be interpreted as mode construction operators and there may not be a Bogoliubov transformation which transforms the Hamiltonian to a sum of mutually commuting quasi-particle number operators (Broadbridge 1979).

In the method of algebraic quantisation due to Segal (1963), classical real phase space $\mathcal{M}$ is treated as complex single-particle Hilbert space, provided there can be found a suitable complex structure $J$ on $\mathcal{M}$ which allows the dynamics $C(t)$ to be viewed as unitary. From single-particle space, one constructs the Fock representation of the CCR by the method of Cook (1953). This may be viewed as the GNS cyclic representation constructed from the unique vacuum expectation functional on a $C^*$-algebra of observables (Segal 1963). However, it was shown (Broadbridge and Hurst 1981b, Broadbridge 1983) that for unstable classical dynamics $C(t)$, no such suitable complex structure exists. This is the algebraic counterpart of the abovementioned no-go theorem, for heuristic quantisation of unstable modes. Therefore, having now discussed the necessity of the indefinite metric, we ask whether algebraic quantisation can be applied to unstable dynamical systems by relaxing the positivity requirement. The main purpose of this paper is to find the most general quadratic Hamiltonian for which the corresponding dynamical group may be considered pseudo-unitary on an indefinite inner product space. We not only achieve this but also produce a pseudo-unitarising complex structure explicitly, whenever one exists. The assumption of pseudo-unitary single-particle dynamics was the starting point of the rigorous Cook-type construction of Fock space with indefinite metric, previously reported by Mintchev (1980). The present paper, along with Mintchev's, constitutes an algebraic quantisation with indefinite metric. There remain serious problems in the physical interpretation of the indefinite metric and these are discussed in § 5, along with some possible resolutions which are still being investigated by other researchers.

## 2. Some necessary conditions for pseudo-unitarisability

In the rigorous method of algebraic quantisation due to Segal (1963), $C(t)$ is treated as a one-parameter group of complex unitary transformations, which preserve the complex inner product

$$\langle z, z'\rangle_1 = -B(z, Jz') - \mathrm{i}B(z, z'), \tag{2.1}$$

with $J$ a real linear transformation of $R^{2N}$ satisfying

$$J^2 = -I \qquad (J \text{ is a complex structure}), \tag{2.2a}$$

$$J^{\mathrm{T}}GJ = G \qquad (J \text{ is symplectic, since } \langle\cdot,\cdot\rangle_1 \text{ must be a } J\text{-sesquilinear form}), \tag{2.2b}$$

$$-GJ > 0 \qquad \text{(positivity of metric } \langle\cdot,\cdot\rangle_1), \tag{2.2c}$$

$$[J, G\hat{H}] = 0 \qquad \text{(since } C(t) \text{ must be unitary).} \tag{2.2d}$$

Provided that such a complex structure $J$ exists, the complex inner product space $\mathcal{H}^{(1)} = (\boldsymbol{R}^{2N}, J, \langle\cdot,\cdot\rangle_1)$ may be viewed as single-particle Hilbert space. In this paper, we shall neglect the positivity condition (2.2$c$).

It has been proven by Tolimieri (1978) and Rossi (1981) that for each signature of $-GJ$, there is a separate unique conjugacy class containing $J$. We proved this independently (Broadbridge and Hurst 1981b), using the theory of symplectic canonical forms (Broadbridge 1979).

*Proposition 2.3.* Every symplectic complex structure $J$ lies in the same $\mathrm{Sp}(2N, \boldsymbol{R})$ conjugacy class as a matrix of the form

$$S = \pm G^{(2)} \oplus_s \pm G^{(2)} \ldots \oplus_s \pm G^{(2)}.$$

Here, the superscript in $G^{(2)}$ denotes the order of the matrix and $\oplus_s$ denotes the symplectic direct sum, which is obtained from the ordinary direct sum by a reordering of the basis

$$(q_1, p_1, q_2, p_2, \ldots, q_m, p_m) \to (q_1, \ldots, q_m, p_1, \ldots, p_m).$$

This reordering ensures that

$$G^{(2k)} \oplus_s G^{(2l)} = G^{(2k+2l)} = \begin{pmatrix} 0 & -I^{(k+l)} \\ I^{(k+l)} & 0 \end{pmatrix}.$$

The proof of proposition 2.3 may be obtained from Broadbridge and Hurst (1981b).

A change of order in the symplectic basis then ensures that the components $+G^{(2)}$ appear first, followed by a string of components $-G^{(2)}$. As a result, there exists a symplectic transformation $C$ such that

$$C^{-1}JC = G^{(2)} \oplus_s \ldots \oplus_s G^{(2)} \oplus_s - G^{(2)} \oplus_s \ldots \oplus_s - G^{(2)}$$

$$= \begin{bmatrix} 0 & 0 & -I^{(a)} & 0 \\ 0 & 0 & 0 & I^{(b)} \\ I^{(a)} & 0 & 0 & 0 \\ 0 & -I^{(b)} & 0 & 0 \end{bmatrix}. \tag{2.4}$$

If a symplectic transformation $C$ commutes with $G(a, b)$ then $C$ is unitary on the complex inner product space

$$\mathcal{H}^{(1)} = (\boldsymbol{R}^{2N}, J, \langle\cdot,\cdot\rangle_1) \qquad \text{where } J = G(a, b)$$

and

$$\langle\cdot,\cdot\rangle_1 = -(\cdot, GJ\,\cdot) - \mathrm{i}(\cdot, G\,\cdot), \tag{2.5}$$

$$-GJ = -GG(a, b) = \mathrm{diag}[I^{(a)}, -I^{(b)}, I^{(a)}, -I^{(b)}]. \tag{2.6}$$

With $J = G(a, b)$, the first $N$ basis vectors of the real space span the whole complex space $\mathcal{H}^{(1)}$, since

$$e_{N+j} = \begin{cases} Je_j & \text{for } j = 1, \ldots, a, \\ -Je_j & \text{for } j = a+1, \ldots, N. \end{cases} \tag{2.7}$$

In the general case that $J$ is conjugate to $G(a, b)$,

$$C^{-1}JC = G(a, b) \quad \text{and} \quad C^{\mathrm{T}}GC = G \tag{2.8}$$

$$\Rightarrow C^{\mathrm{T}}(-GJ)C = -GG(a, b). \tag{2.9}$$

Therefore, if $-GJ$ has signature $(2a, 2b)$, $J$ is conjugate to $G(a, b)$ and by (2.5)–(2.9), the group of real symplectic transformations which commute with $J$ is isomorphic to the group $\mathrm{U}(a, b)$ which preserves a complex inner product $\langle \cdot , \cdot \rangle_1$ of signature $(a, b)$.

*Definition 2.10.* A one-parameter symplectic group $C(t) = \exp(-G\hat{H}t)$ has a pseudo-unitarising complex structure $J$ if $J$ is a symplectic complex structure and $J$ commutes with $G\hat{H}$.

*Proposition 2.11.* $C(t) = \exp(-G\hat{H}t)$ has a pseudo-unitarising complex structure $J$ if and only if for some $C \in \mathrm{Sp}(2N, R)$,

$$C^{\mathrm{T}}\hat{H}C = \hat{H}_0 = \begin{pmatrix} A & LK \\ -KL & KAK \end{pmatrix}$$

with $A = A^{\mathrm{T}}$, $L = -L^{\mathrm{T}}$ and $K = \mathrm{diag}[-I^{(a)}, I^{(b)}]$ with $a + b = N$.

*Proof.* Since $J$ is a symplectic complex structure, by (2.3), there exists $C \in \mathrm{Sp}(2N, R)$ such that $C^{-1}JC = G(a, b) = \left(\begin{smallmatrix} 0 & K \\ -K & 0 \end{smallmatrix}\right)$, with $K$ defined as above. Now

$$\begin{aligned} [J, G\hat{H}] = 0 &\Leftrightarrow [CG(a, b)C^{-1}, G\hat{H}] = 0 \\ &\Leftrightarrow [G(a, b), C^{-1}G\hat{H}C] = 0 \\ &\Leftrightarrow [G(a, b), GC^{\mathrm{T}}\hat{H}C] = 0 \\ &\Leftrightarrow [G(a, b), C^{\mathrm{T}}\hat{H}C] = 0 \qquad (\text{since } [G(a, b), G] = 0). \end{aligned} \tag{2.12}$$

The symmetric matrix $C^{\mathrm{T}}\hat{H}C$ must have the structure $\left(\begin{smallmatrix} A & B \\ B^{\mathrm{T}} & F \end{smallmatrix}\right)$ with $A - A^{\mathrm{T}} = F - F^{\mathrm{T}} = 0$. Then (2.12) is equivalent to

$$-BK = KB^{\mathrm{T}}(= (BK)^{\mathrm{T}}) \tag{2.13a}$$

and

$$AK = KF. \tag{2.13b}$$

Since $K$ is an involution, $F = KAK$ and $B = LK$, where $L = BK$ is skew symmetric, by (2.13$a$).

Conversely, if for some $C \in \mathrm{Sp}(2N, R)$, $C^{\mathrm{T}}\hat{H}C$ has the form given in proposition 2.11,

$$\begin{aligned} &[GC^{\mathrm{T}}\hat{H}C, G(a, b)] = 0 \\ &\qquad \Leftrightarrow [G\hat{H}, CG(a, b)C^{-1}] = 0 \\ &\qquad \Leftrightarrow CG(a, b)C^{-1} \text{ is a pseudo-unitarising complex structure for } \exp(-G\hat{H}t). \end{aligned}$$

*Definition 2.14.* The canonical orbit containing the real symmetric matrix $\hat{H}$ is defined to be the set $\{C^{\mathrm{T}}\hat{H}C ; C \in \mathrm{Sp}(2N, R)\}$.

The canonical orbits are in one-to-one correspondence with the conjugacy classes of the Lie algebra $\mathrm{sp}(2N, R)$, since with $C \in \mathrm{Sp}(2N, R)$, $C^{\mathrm{T}}\hat{H}C = \hat{H}_1 \Leftrightarrow C^{-1}G\hat{H}C = G\hat{H}_1$. From proposition 2.11, existence of $J$ is a property which is determined by the canonical orbit of $\hat{H}$. If $J$ is a symplectic complex structure which commutes with $G\hat{H}$ and if $\hat{H}_1 = C^{\mathrm{T}}\hat{H}C$, with $C \in \mathrm{Sp}(2N, R)$, then it is easy to verify that $\exp(-G\hat{H}_1 t)$ has a pseudo-unitarising complex structure

$$J_1 = C^{-1}JC. \tag{2.15}$$

To construct a solution $J$ in the most general case, we need only consider a canonical form belonging to each canonical orbit and then apply (2.15). The full set of canonical forms was first given by Williamson (1936). In the notation which we have used repeatedly (e.g. Broadbridge 1979), the indecomposable canonical forms are denoted $\hat{K}_j^{(2k)}(\rho, s_j)$. Associated invariants include the elementary divisors $(s \pm s_j)^k$ for the pencil $(\mathrm{i}G\hat{H} - sI)$. $\rho = \pm 1$ distinguishes two possible distinct orbits when $j = 4$, 5 or 6, in which cases the frequencies $s_j$ are real. When $j = 4$ or 5, $\rho$ is determined by the signature of $\hat{H}$. For $j = 6$, the case of real frequencies and Jordan chains of even length $k$, $\rho$ is determined by the signature of $\hat{H}_N$, $-G\hat{H}_N$ being the nilpotent part in the Jordan decomposition of $-G\hat{H}$ (Cushman 1973). In the other three cases, $j = 1$, 2 or 3, the elementary divisors determine the canonical orbit.

Even after allowing $\mathscr{H}^{(1)}$ to have indefinite metric, it soon becomes apparent that a pseudo-unitarising complex structure does not always exist. For example, in one degree of freedom every symplectic complex structure $J$ is conjugate to $G(1, 0) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = G^{(2)}$ or to $G(0, 1) = -G^{(2)}$. Therefore, it is clear, from the argument of Gallone and Sparzani (1979), that if $[J, GH] = 0$, $\hat{H}$ must either be trivial ($\hat{H} = 0$) or must belong to the harmonic oscillator class. Neither the single repulsive oscillator ($\hat{H} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$) nor the single free particle ($\hat{H} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$) admit pseudo-unitarisation. In fact, we shall prove that each elementary divisor $(s - \mathrm{i}s_j)^{N_j}$ associated with an imaginary or zero frequency $\mathrm{i}s_j$ must occur an even number of times before the symplectic dynamics in $N$ degrees of freedom can be pseudo-unitarised.

*Proposition 2.16.* Suppose that a pseudo-unitarising complex structure exists for $C(t) = \exp(-G\hat{H}t)$. Then, for any eigenvalue $\mathrm{i}s_j$ of $\mathrm{i}G\hat{H}$ with $s_j \in R$, each elementary divisor $(s - \mathrm{i}s_j)^{N_j}$, for fixed $N_j$, must occur an even number of times.

*Proof.* Let $(s - \mathrm{i}s_j)^{N_j}$, with $s_j \in R$, be an elementary divisor of $\mathrm{i}G\hat{H} - sI$. Since the elementary divisors are invariant under symplectic transformations, they are also the elementary divisors of $\mathrm{i}G\hat{H}_0 - sI$, with $\hat{H}_0$ the canonical form of (2.11). By the Jordan canonical form theorem, there exists a basis of root vectors $\boldsymbol{e}_{j,l}$ satisfying

(a) $$(\mathrm{i}G\hat{H}_0 - \mathrm{i}s_j I)\boldsymbol{e}_{j,l} = \begin{cases} \mathbf{0} & \text{if } l = N_j, \\ \boldsymbol{e}_{j,l+1} & \text{if } l < N_j, \end{cases}$$

(b) $$\boldsymbol{e}_{j,1} \in \mathrm{ran}(\mathrm{i}G\hat{H}_0 - \mathrm{i}s_j I).$$

Let $\Gamma$ be the antilinear operator on $C^{2N}$ defined by

$$\Gamma\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{u}^* \\ \boldsymbol{v}^* \end{pmatrix} \qquad (K \text{ as in } (2.11)), \tag{2.17}$$

with $\boldsymbol{u}^*$ the complex conjugate of $\boldsymbol{u}$. Then

$$(\mathrm{i}G\hat{H}_0 - \mathrm{i}s_j I)\Gamma \boldsymbol{e}_{j,l} = \begin{cases} \boldsymbol{0} & \text{if } l = N_j, \\ -\Gamma \boldsymbol{e}_{j,l+1} & \text{if } L < N_j. \end{cases}$$

Therefore, the vectors $(-1)^l \Gamma \boldsymbol{e}_{j,l}$, $l = 1, \ldots, N_j$, constitute another Jordan basis corresponding to the elementary divisor $(s - \mathrm{i}s_j)^{N_j}$. Furthermore, $\{\boldsymbol{e}_{j,l}\}_{l=1,\ldots,N_j} \cup \{\Gamma \boldsymbol{e}_{j,l}\}_{l=1,\ldots,N_j}$ is a set of $2N_j$ linearly independent vectors, since if

$$\boldsymbol{0} = \sum_{l=1}^{N_j} \alpha_l \boldsymbol{e}_l + \beta_l \Gamma \boldsymbol{e}_l \tag{2.18}$$

(temporarily neglecting the suffix $j$), by applying $(\mathrm{i}G\hat{H}_0 - \mathrm{i}s_j I)^{N_j - 1}$ to each side of (2.18), we obtain

$$\boldsymbol{0} = \alpha_1 \boldsymbol{e}_{N_j} - (-1)^{N_j} \beta_1 \Gamma \boldsymbol{e}_{N_j}.$$

If $\beta_1 \neq 0$, then this implies

$$\Gamma \boldsymbol{e}_{N_j} = \alpha \boldsymbol{e}_{N_j} \qquad \text{with } \alpha = (-1)^{N_j} \alpha_1 / \beta_1. \tag{2.19}$$

The vector $\boldsymbol{e}_{N_j}$ in $C^{2N}$ may be expressed as $\binom{\boldsymbol{u}}{\boldsymbol{v}}$. Then (2.19) can be re-expressed as

$$\boldsymbol{K}\boldsymbol{v}^* = \alpha \boldsymbol{u} \qquad \text{and} \qquad -\boldsymbol{K}\boldsymbol{u}^* = \alpha \boldsymbol{v} \Rightarrow \boldsymbol{v}^* = \alpha \boldsymbol{K}\boldsymbol{u} \qquad (\text{since } \boldsymbol{K}^2 = I)$$

and

$$\begin{aligned} \boldsymbol{u} = -\alpha^* \boldsymbol{K}\boldsymbol{v}^* \Rightarrow \boldsymbol{v}^* &= -|\alpha|^2 \boldsymbol{v}^* \\ &\Rightarrow \boldsymbol{v} = 0 \\ &\Rightarrow \boldsymbol{v} = 0 \qquad \text{and} \qquad \boldsymbol{u} = 0. \end{aligned}$$

Since $\boldsymbol{e}_{N_j}$ is non-null, this is a contradiction and so we deduce $\alpha_1 = \beta_1 = 0$. Assuming $\alpha_1 = \beta_1 = 0$, applying $(\mathrm{i}G\hat{H}_0 - \mathrm{i}s_j I)^{N_j - 2}$ to each side of (2.18), we similarly obtain $\alpha_2 = \beta_2 = 0$. Continuing this recursive attack, we can show $\alpha_l = \beta_l = 0$ for all $l$. Therefore, the vectors $\{\boldsymbol{e}_l\}$ and $\{\Gamma \boldsymbol{e}_l\}$ are linearly independent.

Let us define $V_j = \mathrm{span}\{\boldsymbol{e}_{j,l}, \Gamma \boldsymbol{e}_{j,l};\, l = 1, \ldots, N_j\}$ and define the skew-orthogonal complement

$$V_j^\perp = \{\boldsymbol{w} \in C^{2N};\, B(\boldsymbol{w}, \boldsymbol{v}) = 0 \text{ for all } \boldsymbol{v} \in V_j\}.$$

It is not difficult to see that $V_j^\perp$ is not only $\mathrm{i}G\hat{H}$-invariant but also $\Gamma$-invariant. Therefore, if $\mathrm{i}s_k$ is another pure imaginary frequency, we may similarly extract a subspace $V_k = \mathrm{span}\{\boldsymbol{e}_{k,l}, \Gamma \boldsymbol{e}_{k,l};\, l = 1, \ldots, N_k\}$ from $V_j^\perp$, and on this subspace, $(\mathrm{i}G\hat{H}_0 - sI)$ has a pair of elementary divisors $(s - \mathrm{i}s_k)^{N_k}$. We may continue to extract pairs of Jordan bases until all imaginary frequencies have been accounted for. This completes the proof of proposition 2.16.

When looking for the full set of Hamiltonians for which $J$ exists, it is natural to enquire whether the condition given in proposition 2.16 is sufficient as well as necessary. The following result will help to produce counterexamples to this conjecture.

*Proposition 2.20.* Suppose that a pseudo-unitarising complex structure exists for $C(t) = \exp(-G\hat{H}t)$. Then $\mathrm{sig}\, \hat{H} = (0, 0)$ (modulo 2).

*Proof.* Suppose that a pseudo-unitarising complex structure exists for $C(t) = \exp(-G\hat{H}t)$. Then, by proposition 2.11, there exists $C \in \mathrm{Sp}(2N, R)$ such that $C^{\mathrm{T}}\hat{H}C = \left(\begin{smallmatrix} A & LK \\ -KL & KAK \end{smallmatrix}\right)$ with $A = A^{\mathrm{T}}$, $L = -L^{\mathrm{T}}$ and $K = \mathrm{diag}[-I^{(a)}, I^{(b)}]$ with $a + b = N$:

$$C^{\mathrm{T}}\hat{H}C = \begin{pmatrix} I & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} A & L \\ -L & A \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & K \end{pmatrix}.$$

Since $\left(\begin{smallmatrix} I & 0 \\ 0 & K \end{smallmatrix}\right)$ is an involution, $\hat{H}$ must have the same signature as $\left(\begin{smallmatrix} A & L \\ -L & A \end{smallmatrix}\right)$. This implies that $\hat{H}$ has the same signature as

$$P^{-1}\begin{pmatrix} A & L \\ -L & A \end{pmatrix} P = \begin{pmatrix} A - \mathrm{i}L & 0 \\ 0 & A + \mathrm{i}L \end{pmatrix} \qquad \left(\text{with } P = 2^{-1/2}\begin{pmatrix} I & I \\ -\mathrm{i}I & \mathrm{i}I \end{pmatrix} = P^{\dagger -1}\right).$$

Therefore,

$$\begin{aligned} \operatorname{sig} \hat{H} &= \operatorname{sig}(A - \mathrm{i}L) + \operatorname{sig}(A + \mathrm{i}L) = \operatorname{sig}(A + \mathrm{i}L)^* + \operatorname{sig}(A + \mathrm{i}L) \\ &= 2 \operatorname{sig}(A + \mathrm{i}L) \qquad (\text{since } A + \mathrm{i}L \text{ is Hermitian}). \end{aligned}$$

Hence, both the positive and negative eigenspaces of $\hat{H}$ must be even dimensional.

As an application of proposition 2.20, we recall that $\hat{K}_3^{(2k)}$ has signature $(k-1, k-1)$. Therefore, if $\hat{H}$ belongs to the same canonical class as $\hat{K}_3^{(2k)}$, with $k$ even, $\exp(-G\hat{H}t)$ cannot be pseudo-unitarised, even though the elementary divisors of $\mathrm{i}G\hat{H} - sI$ occur in pairs $s^k, s^k$. This provides a counterexample for the converse of (2.16).

As another example, let $\hat{H} = \hat{K}_4^{(2k)}(\rho_1) \oplus_{\mathrm{s}} \hat{K}_4^{(2k)}(\rho_2)$. $\mathrm{i}G\hat{H} - sI$ has a pair of elementary divisors $s^{2k}, s^{2k}$ but $\operatorname{sig} \hat{H}$ may be varied by varying $\rho_1 (= \pm 1)$ and $\rho_2 (= \pm 1)$. Since $\ker \hat{K}_4^{(2k)}(\rho_1)$ is one dimensional, $\operatorname{sig} \hat{K}_4^{(2k)}(\rho_1) = (0, 1)$ or $(1, 0)$ (modulo 2). We recall from Broadbridge (1979) that $\hat{K}_4^{(2k)}(-\rho_1)$ may be obtained from $\hat{K}_4^{(2k)}(\rho_1)$ by a signature-reversing transformation $\hat{K}_4^{(2k)}(-\rho_1) = -\left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right) \hat{K}_4^{(2k)}(\rho_1) \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. Therefore, $\operatorname{sig} \hat{K}_4^{(2k)}(-\rho_1) = (1, 1) - \operatorname{sig} \hat{K}_4^{(2k)}(\rho_1)$ (modulo 2). This implies that if $\rho_2 = -\rho_1$,

$$\operatorname{sig}(\hat{K}_4^{(2k)}(\rho_1) \oplus_{\mathrm{s}} \hat{K}_4^{(2k)}(\rho_2)) = (1, 1) \text{ (modulo 2)}.$$

Therefore, by proposition 2.20, if $\hat{H}$ belongs to the same canonical class as $\hat{K}_4^{(2k)}(\rho_1) \oplus_{\mathrm{s}} \hat{K}_4^{(2k)}(\rho_2)$, $\exp(-G\hat{H}t)$ cannot be pseudo-unitarised unless $\rho_1 = \rho_2$. It is not good enough that the elementary divisors occur in pairs $s^{2k}$, $s^{2k}$. The other invariants $\rho_1$ and $\rho_2$ on the associated principal subspaces must also be equal.

## 3. Appropriate complex structure

From § 2, a canonical form containing an odd number of identical components $\hat{K}_1^{(2k)}(a\mathrm{i})$ or an odd number of identical components $\hat{K}_4^{(2k)}(\rho)$ or $\hat{K}_3^{(4k)}$ indicates that no appropriate complex structure $J$ exists. In all other cases, the dynamics can be pseudo-unitarised, which we shall demonstrate by constructing $J$ explicitly.

We shall begin with the canonical system possessing complex frequencies, which leads to some canonical systems with pure imaginary or zero frequencies as special cases. On a $G\hat{H}$-invariant subspace on which $\hat{H}$ is $\hat{K}_2^{(4k)}(b + a\mathrm{i})$, one solution for $J$ in (2.2$a$, $b$, $d$) is

$$J = \pm \mathrm{diag}[G^{(2)}, G^{(2)}, \ldots, G^{(2)}]. \tag{3.1}$$

The case $\hat{H} = \hat{K}_2^{(4k)}(b+a\mathrm{i})$, for which $\mathrm{i}G\hat{H} - sI$ has elementary divisors $(s+b+a\mathrm{i})^k$, $(s+b-a\mathrm{i})^k$, $(s-b+a\mathrm{i})^k$ and $(s-b-a\mathrm{i})^k$, reduces to the canonical class of $\hat{H} = \hat{K}_1^{(2k)} \oplus_{\mathrm{s}} \hat{K}_1^{(2k)}$, when $b = 0$. The canonical form $\hat{K}_1^{(4k)}(a\mathrm{i}) \oplus_{\mathrm{s}} \hat{K}_1^{(4k)}(a\mathrm{i})$ reduces directly to $\hat{K}_3^{(4k)} \oplus_{\mathrm{s}} \hat{K}_3^{(4k)}$, when $a = 0$. Therefore, the latter examples admit a complex structure given in (3.1). Now we shall consider the special case $\hat{H} = \hat{K}_3^{(2k)}$ with $k$ odd. In the simplest case, $k = 1$, $\hat{K}_3^{(2)} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $-G\hat{K}_3^{(2)}$ commutes with every symplectic complex structure. With $k = 3$, the conditions $J = -I$, $J^{\mathrm{T}}GJ = G$ and $[J, GK_3^{(6)}] = 0$ yield a general solution for the $6\times 6$ matrix $J$:

$$J = \begin{bmatrix} \alpha & 0 & \beta & \gamma & 0 & -\delta \\ 0 & \alpha & 0 & 0 & \delta & 0 \\ 0 & 0 & \alpha & -\delta & 0 & 0 \\ 0 & 0 & \varepsilon & -\alpha & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & -\alpha & 0 \\ \varepsilon & 0 & \mu & -\beta & 0 & -\alpha \end{bmatrix}$$

with

$$\delta \neq 0, \qquad \varepsilon = (1+\alpha^2)/\delta, \qquad \mu = 2\alpha\beta/\delta + \gamma(1+\alpha^2)/\delta^2, \qquad \alpha, \beta, \gamma, \delta, \varepsilon, \mu \in R. \tag{3.2}$$

The example

$$J = \begin{bmatrix} & & & \cdot & 1 & 0 & -1 \\ & & & \cdot & 0 & 1 & 0 \\ & & & \cdot & -1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & & & \\ 0 & -1 & 0 & \cdot & & & \\ 1 & 0 & 1 & \cdot & & & \end{bmatrix}$$

extends to a pseudo-unitarising complex structure for the general case $\hat{K}_3^{(2k)}$ with $k$ odd:

$$J = \begin{bmatrix} & & & & & & & \cdot & & & & & 1 & 0 & -1 \\ & & & & & & & \cdot & & & & -1 & 0 & 1 & \\ & & & & & & & \cdot & & & \cdot & \cdot & -1 & & \\ & & & & & & & \cdot & & -1 & \cdot & \cdot & & & \\ & & & & & & & \cdot & 1 & 0 & \cdot & & & & \\ & & & & & & & \cdot & 0 & 1 & & & & & \\ & & & & & & & \cdot & -1 & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & 1 & \cdot & & & & & & & \\ & & & & & -1 & 0 & \cdot & & & & & & & \\ & & & & \cdot & 0 & 1 & \cdot & & & & & & & \\ & & & \cdot & \cdot & -1 & & \cdot & & & & & & & \\ & & & & & \cdot & & \cdot & & & & & & & \\ & & \cdot & \cdot & \cdot & & & \cdot & & & & & & & \\ & -1 & 0 & -1 & & & & \cdot & & & & & & & \\ 1 & 0 & 1 & & & & & \cdot & & & & & & & \end{bmatrix} \tag{3.3}$$

For the case $\hat{H} = \hat{K}_4^{(2k)}(\rho) \oplus_s \hat{K}_4^{(2k)}(\rho)$, one solution is

$$J = \pm \operatorname{diag}[G^{(2k)}, G^{(2k)}]. \tag{3.4}$$

For $\hat{H} = \hat{K}_5^{(4k+2)}(\rho, b)$ and $\hat{H} = K_6^{(4k)}(\rho, b)$, we may obtain a solution by first transforming $\hat{H}$ to the canonical form of proposition 2.11. Our results are that with $\hat{H} = \hat{K}_5^{(4k+2)}(\rho, b)$, one solution is

$$J = \left[\begin{array}{ccccccc|ccccccc}
 & & & & & & & & & & & & & -b \\
 & & & & & & & & & & & & \cdot & \\
 & & & & & & & & & & & b & & \\
 & & & & & & & & & & -b & & & \\
 & & & & & & & & & \cdot & & & & \\
 & & & & & & & & b & & & & & \\
 & & & & & & & -b & & & & & & \\
\hline
 & & & & & & b^{-1} & & & & & & & \\
 & & & & & -b^{-1} & & & & & & & & \\
 & & & & \cdot & & & & & & & & & \\
 & & & \cdot & & & & & & & & & & \\
 & & \cdot & & & & & & & & & & & \\
 & -b^{-1} & & & & & & & & & & & & \\
b^{-1} & & & & & & & & & & & & &
\end{array}\right] \tag{3.5}$$

and with $\hat{H} = \hat{K}_6^{(4k)}(\rho, b)$, one solution is

$$J = \left[\begin{array}{cccccccccc}
0 & -b & & & & & & & & \\
b^{-1} & 0 & & & & & & & & \\
 & & \ddots & & & & & & & \\
 & & & 0 & -b & & & & & \\
 & & & b^{-1} & 0 & & & & & \\
 & & & & & 0 & -b^{-1} & & & \\
 & & & & & b & 0 & & & \\
 & & & & & & & \ddots & & \\
 & & & & & & & & 0 & -b^{-1} \\
 & & & & & & & & b & 0
\end{array}\right]. \tag{3.6}$$

Having reduced an arbitrary real symmetric $\hat{H}$ to a symplectic direct sum of canonical matrices $K_j^{(2k)}$, and provided that the illegal combinations mentioned at the beginning of this section do not occur, $J$ may be taken to be the symplectic direct sum of the solutions, listed above, for each contributing $G\hat{H}$-invariant subspace. For a solution $J$ when $\hat{H}$ is not canonical, we need only apply (2.15).

## 4. Fock space with indefinite metric

Given a complex (single-particle) Hilbert space $\mathscr{H}^{(1)}$, the rigorous construction of both Fock space $\mathscr{F}(\mathscr{H}^{(1)})$ over $\mathscr{H}^{(1)}$ and the natural creation and annihilation operators of

$\mathscr{F}(\mathscr{H}^{(1)})$ were devised by Cook (1953). The Segal procedure for quantisation of linear systems, which we have now fully investigated, involves finding a complex structure $J$ on the classical real linear symplectic space $\mathscr{M}=(R^{2N},B)$ so that the classical symplectic dynamics $C(t)$ is unitary with respect to the associated $J$-sesquilinear form $\langle\cdot,\cdot\rangle_1$. The complex space $(\mathscr{M},J,\langle\cdot,\cdot\rangle_1)$ then becomes $\mathscr{H}^{(1)}$ in the Cook construction. Mintchev (1980) has extended the Cook construction to the case that $\mathscr{H}^{(1)}$ is an indefinite inner product space. The results which we have presented so far in this section may be regarded as a rigorous construction of a complex indefinite inner product space $\mathscr{H}^{(1)}$ with pseudo-unitary dynamics, which was the starting point of Mintchev's work. Given an indefinite inner product $\langle\phi,\psi\rangle_1=(\phi,\eta\psi)$, with $\eta$ a self-adjoint contraction on complex Hilbert space $(\mathscr{H}(\cdot,\cdot))$, define $\mathscr{H}^{(1)}=(\mathscr{H},\langle\cdot,\cdot\rangle_1)$. Fock space $\mathscr{F}(\mathscr{H}^{(1)})$ with indefinite inner product $\langle\cdot,\cdot\rangle$ is constructed as in the usual Cook construction:

$$\mathscr{F}(\mathscr{H}^{(1)})=\bigoplus_{n=0}^{\infty}\mathscr{H}^{(n)} \tag{4.1}$$

where $\mathscr{H}^{(0)}=C$ and the space $\mathscr{H}^{(n)}$ includes all symmetrised tensor products $S_n\phi_1\otimes\ldots.\otimes\phi_n$ of $n$ vectors of $\mathscr{H}^{(1)}$. Here, $S_n$ is the symmetrisation operator

$$S_n=\frac{1}{n!}\sum_{\sigma\in\mathrm{P}_n}\sigma,$$

where $\mathrm{P}_n$ is the symmetric group of permutations of $n$ symbols. Annihilation operators are defined by

$$a(\phi)S_n\phi_1\otimes\ldots\otimes\phi_n=n^{1/2}S_n\langle\phi,\phi_1\rangle_1\phi_2\otimes\ldots\otimes\phi_n \qquad \text{for } n\geqslant 1$$

and

$$a(\phi)(\mathscr{H}^{(0)})=\{0\}. \tag{4.2}$$

The pseudo-adjoint $a^+(\phi)$ of $a(\phi)$ behaves like a creation operator

$$a^+(\phi_{n+1})S_n\phi_1\otimes\ldots\otimes\phi_n=(n+1)^{1/2}S_{n+1}\phi_1\otimes\ldots\otimes\phi_n\otimes\phi_{n+1}. \tag{4.3}$$

Field operators $\Phi(\psi)$ are defined by

$$\Phi(\psi)=2^{-1/2}(a(\psi)+a^+(\psi)). \tag{4.4}$$

We write the finite-particle subspace as

$$\mathscr{F}_0=\{\phi\in\mathscr{F}(\mathscr{H}^{(1)});\ \exists n\in Z,\phi=(\phi^{(0)},\phi^{(1)},\ldots,\phi^{(m)},\ldots)\text{ with }\phi^{(m)}=0\text{ for all }m>n\}.$$

*Theorem 4.5.* (Mintchev 1980)

(a) $\Phi(\phi)$ is closable for all $\phi\in\mathscr{H}^{(1)}$.

(b) $\mathscr{F}_0$ is a set of analytic vectors for $\Phi(\phi)$, for all $\phi\in\mathscr{H}^{(1)}$.

(c) If $\{\phi_k\}\subset\mathscr{H}^{(1)}$ and $\operatorname{s-lim}_{k\to\infty}\phi_k=\phi$, then

$$\operatorname*{s-lim}_{k\to\infty}\Phi(\phi_k)\psi=\Phi(\phi)\psi \text{ for all } \psi\in\mathscr{F}_0.$$

(d) The vacuum vector $\psi_0$ is cyclic with respect to $\{\Phi(\phi);\ \phi\in\mathscr{H}^{(1)}\}$.

(e) For all $\chi\in\mathscr{F}_0$ and $\phi,\psi\in\mathscr{H}^{(1)}$, $[\Phi(\phi),\Phi(\psi)]\chi=\mathrm{i}\operatorname{Im}\langle\phi,\psi\rangle_1\chi$ and $[a(\phi),a^+(\psi)]\chi=\langle\phi,\psi\rangle_1\chi$.

Since the operators $\Phi(\psi)$ satisfy the CCR on $\mathscr{F}_0$, which is a subspace of analytic vectors for $\Phi(\psi)$, just as in the case of positive metric, the first similarity rule of Friedrichs (1953) would be valid, so that

$$a^+(\boldsymbol{\xi}_1(t)) \ldots a^+(\boldsymbol{\xi}_n(t))\psi_0 = \mathrm{e}^{-\mathrm{i}(H+\alpha)t} a^+(\boldsymbol{\xi}_1(0)) \ldots a^+(\boldsymbol{\xi}_n(0))\psi_0, \tag{4.6}$$

where

$$H = \tfrac{1}{2}\boldsymbol{Z}^{\mathrm{T}}\hat{H}\boldsymbol{Z}, \qquad Z_\mu = \sum_{\nu=1}^{2N} G_{\mu\nu}\Phi(\boldsymbol{e}_\nu) \qquad (Z_\mu \text{ obey the CCR}) \tag{4.7}$$

and in (4.6) it is assumed that $H\psi_0 = \alpha\psi_0$ with $\alpha \in R$. This assumption is valid when $\hat{H}$ is reduced to the canonical form $\hat{H}_0$ of proposition (2.9) and $J = G(a, b)$. In this case,

$$H\psi_0 = \tfrac{1}{2} \sum_{\mu,\nu=1}^{2N} \Phi(\boldsymbol{e}_\mu)\hat{Y}_{\mu\nu}\Phi(\boldsymbol{e}_\nu)\psi_0, \tag{4.8}$$

where

$$\hat{Y} = G^{\mathrm{T}}\hat{H}_0 G = \begin{pmatrix} KAK & KB \\ -BK & A \end{pmatrix}, \qquad A = A^{\mathrm{T}}, \qquad B = -B^{\mathrm{T}},$$

$$K = \mathrm{diag}[\rho_1, \ldots, \rho_N] \qquad \text{with } \rho_j = \begin{cases} -1 & \text{if } j \leqslant a, \\ +1 & \text{if } j > a. \end{cases}$$

Since $a(J\boldsymbol{\xi}) = -\mathrm{i}a(\boldsymbol{\xi})$ and $J = G(a, b)$, we have

$$a(\boldsymbol{e}_{j+N}) = \begin{cases} a(J\boldsymbol{e}_j) = -\mathrm{i}a(\boldsymbol{e}_j) & \text{if } 1 \leqslant j \leqslant a, \\ a(-J\boldsymbol{e}_j) = \mathrm{i}a(\boldsymbol{e}_j) & \text{if } a+1 \leqslant j \leqslant N. \end{cases} \tag{4.9}$$

Combining (4.9) with (4.4), we obtain

$$\begin{aligned} &\Phi(\boldsymbol{e}_j) = 2^{-1/2}[a(\boldsymbol{e}_j) + a^+(\boldsymbol{e}_j)], \\ &\Phi(\boldsymbol{e}_{j+N}) = 2^{-1/2}[\mathrm{i}\rho_j a(\boldsymbol{e}_j) - \mathrm{i}\rho_j a^+(\boldsymbol{e}_j)], \qquad \text{for all } j \leqslant N. \end{aligned} \tag{4.10}$$

We shall write (4.10), in a condensed notation, as

$$\boldsymbol{\Phi} = P'\boldsymbol{\alpha} \tag{4.11}$$

with

$$\alpha_j = a(\boldsymbol{e}_j), \qquad \alpha_{j+N} = a^+(\boldsymbol{e}_j), \qquad \Phi_\mu = \Phi(\boldsymbol{e}_\mu), \qquad P' = 2^{-1/2}\begin{pmatrix} I & I \\ \mathrm{i}K & -\mathrm{i}K \end{pmatrix}.$$

Substituting (4.10) into (4.8), we obtain

$$\begin{aligned} H\psi_0 &= \tfrac{1}{2}\boldsymbol{\Phi}^{\mathrm{T}}\hat{Y}\boldsymbol{\Phi}\psi_0 = \tfrac{1}{2}\boldsymbol{\Phi}^+\hat{Y}\boldsymbol{\Phi}\psi_0 \qquad (\text{since } \Phi_\mu^+ = \Phi_\mu \text{ on } \mathscr{F}_0) \\ &= \tfrac{1}{2}(P'\boldsymbol{\alpha})^+ Y(P'\boldsymbol{\alpha})\psi_0 \\ &= \tfrac{1}{2}\boldsymbol{\alpha}^+ P'^+\hat{Y}P'\boldsymbol{\alpha}\psi_0 \qquad (\text{where } \boldsymbol{\alpha}^+ = (\alpha_1^+, \ldots, \alpha_N^+, \alpha_1, \ldots, \alpha_N)). \end{aligned} \tag{4.12}$$

Now

$$P'^+\hat{Y}P' = \begin{pmatrix} K(A+\mathrm{i}B)K & 0 \\ 0 & K(A-\mathrm{i}B)K \end{pmatrix}. \tag{4.13}$$

Therefore, from (4.12),

$$H\psi_0 = \tfrac{1}{2}\sum_{j,k=1}^{N} (K(A-\mathrm{i}B)K)_{jk} a(\boldsymbol{e}_j) a(\boldsymbol{e}_k)^+ \psi_0$$

$$= \tfrac{1}{2}\sum_{j,k=1}^{N} (K(A-\mathrm{i}B)K)_{jk} \langle \boldsymbol{e}_j, \boldsymbol{e}_k \rangle_1 \psi_0$$

$$= \alpha\psi_0 \qquad \text{where } \alpha = -\tfrac{1}{2}\sum_{j=1}^{N} \rho_j A_{jj} = -\tfrac{1}{2}\mathrm{Tr}(KA). \tag{4.14}$$

It will become evident that complex indefinite inner product space $\mathscr{H}^{(1)}$ is a rigorous version of the heuristic single-mode space, which, by theorem 4.30 of Broadbridge (1981), must have indefinite inner product when the classical dynamics is unstable. To clarify this point, we shall continue to consider a classical system with finite degrees of freedom. Proposition 2.11 then allows us to assume $\hat{H} = \hat{H}_0$ and $G = G(a, b)$, which can always be achieved by a symplectic transformation, given that a pseudo-unitarising complex structure exists. The CCR may then be expressed

$$[a_j, a_k^+] = -\rho_j \delta_{jk} \qquad \text{(from (4.5}e\text{))}$$

and

$$[a_j, a_k] = 0. \tag{4.15}$$

(Here and in the following $a_j$ is an abbreviation for $a(\boldsymbol{e}_j)$.)

To find the single-mode eigenstates for $H$, suppose that $H\phi = \omega\phi$, with $\omega \in C$ and $\phi = \sum_{j=1}^{N} \beta_j a_j^+ \psi_0$. Then

$$\left[H, \sum_{j=1}^{N} \beta_j a_j^+\right]\psi_0 = (\omega - \alpha)\phi.$$

By (4.15) and (4.12)–(4.13), this is equivalent to

$$-\sum_{r,s=1}^{N} \rho_r (A+\mathrm{i}L)_{rs} \beta_s a_r^+ \psi_0 = \sum_{r=1}^{N} (\omega-\alpha)\beta_r a_r^+ \psi_0$$

$$\Leftrightarrow -K(A+\mathrm{i}L)\boldsymbol{\beta} = (\omega-\alpha)\boldsymbol{\beta} \text{ (since the single-mode states } a_r^+\psi_0 \text{ are linearly independent).} \tag{4.16}$$

That is, the vector $\boldsymbol{\beta}$ with $N$ components $\beta_j$ is an eigenvector of $-K(A+\mathrm{i}L)$ corresponding to eigenvalue $\omega - \alpha$. To relate these values to the classical frequencies, notice that

$$\mathrm{i}G\hat{H}_0 = \begin{pmatrix} \mathrm{i}KL & -\mathrm{i}KAK \\ \mathrm{i}A & \mathrm{i}LK \end{pmatrix},$$

which is similar to

$$\begin{pmatrix} I & -\mathrm{i}K \\ I & \mathrm{i}K \end{pmatrix} \mathrm{i}G\hat{H}_0 \begin{pmatrix} I & -\mathrm{i}K \\ I & \mathrm{i}K \end{pmatrix}^{-1} = \tfrac{1}{2}\begin{pmatrix} I & -\mathrm{i}K \\ I & \mathrm{i}K \end{pmatrix} \mathrm{i}G\hat{H}_0 \begin{pmatrix} I & I \\ \mathrm{i}K & -\mathrm{i}K \end{pmatrix}$$

$$= \begin{pmatrix} K(A-\mathrm{i}L) & 0 \\ 0 & -K(A+\mathrm{i}L) \end{pmatrix}.$$

Therefore, the classical frequencies include both the single-mode energies $\omega - \alpha$, which are eigenvalues of $-K(A+\mathrm{i}L)$, and also the values $-(\omega-\alpha)^*$, which are eigenvalues of $K(A-\mathrm{i}L)$. In the most familiar case, $\hat{H}$ is positive definite, and by a theorem of Whittaker (1959), $\hat{H}$ can be reduced to diag$[s_1, \ldots, s_N, s_1, \ldots, s_N]$. This conforms to the canonical matrix $\hat{H}_0$ of proposition 2.11. We may take $A = \mathrm{diag}[s_1, \ldots, s_N]$, $B = 0$ and $K = \mathrm{diag}[-I^{(a)}, I^{(b)}]$, with $a$ arbitrary. The choice $a, b \neq N$ and $J = G(a, b)$ leads to an indefinite inner product space $\mathscr{H}^{(1)}$. However, it is customary to construct a positive definite inner product space whenever possible. This choice, $K = -I$, leads to a set of single-particle energies which includes the eigenvalues $s_j$ of $-K(A + \mathrm{i}L)$ but not the negative eigenvalues $-s_j$ of $\mathrm{i}G\hat{H}$.

For the purposes of this section, we are most interested in the situation when $\mathscr{H}^{(1)}$ may have an indefinite inner product but may not have a definite inner product. For example, consider the case that $\mathrm{i}G\hat{H}$ has simple complex eigenvalues $\pm b \pm a\mathrm{i}$. $\hat{H}$ belongs to the same canonical orbit as

$$\hat{K}_2^{(4)} = \begin{bmatrix} 0 & 0 & -a & -b \\ 0 & 0 & b & -a \\ -a & b & 0 & 0 \\ -b & -a & 0 & 0 \end{bmatrix}.$$

By applying the symplectic transformation

$$C = 2^{-1/2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

$\hat{K}_2^{(4)}$ transforms to

$$C^{\mathrm{T}} \hat{K}_2^{(4)} C = \begin{pmatrix} A & 0 \\ 0 & KAK \end{pmatrix} = \hat{H}_0,$$

with

$$A = \begin{pmatrix} -b & a \\ a & -b \end{pmatrix} = A^{\mathrm{T}} \qquad \text{and} \qquad K = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then according to (4.16), relative to the vacuum, the single-mode energies are the eigenvalues of $-KA$, namely $-b \pm a\mathrm{i}$. The most general single-mode state corresponding to $E = -b + a\mathrm{i}$ is

$$\psi_E = \gamma(a_1^+ \psi_0 + \mathrm{i} a_2^+ \psi_0) \qquad \text{with } \gamma \in C. \tag{4.17a}$$

The most general single-mode state corresponding to $E^* = -b - a\mathrm{i}$ is

$$\psi_{E^*} = \beta(a_1^+ \psi_0 - \mathrm{i} a_2^+ \psi_0) \qquad \text{with } \beta \in C. \tag{4.17b}$$

From the commutation relations (4.5e),

$$\langle \psi_E, \psi_E \rangle = \langle \psi_{E^*}, \psi_{E^*} \rangle = 0 \qquad \text{and} \qquad \langle \psi_{E^*}, \psi_E \rangle = 2\beta^* \gamma. \tag{4.18}$$

From our discussion of heuristic mode space (Broadbridge 1981), we know that the indefinite metric blossoms not only when the classical frequencies are complex, but also when real classical frequencies are non-simple. For example, if $\hat{H}$ belongs to the same canonical orbit as

$$\hat{K}_6^{(4)}(1,b)=\begin{bmatrix} b^{-2} & 0 & 0 & 1 \\ 0 & 1 & -b^2 & 0 \\ 0 & -b^2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

then $\hat{H}$ also belongs to the same canonical orbit as

$$\hat{H}_0=\begin{pmatrix} A & 0 \\ 0 & KAK \end{pmatrix} \qquad \text{with } A=\tfrac{1}{2}\begin{pmatrix} 2b+b^{-1} & -b^{-1} \\ -b^{-1} & -2b+b^{-1} \end{pmatrix} \qquad \text{and}$$

$$K=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.19}$$

$\hat{H}_0=C^{\mathrm{T}}\hat{K}_6^{(4)}C$, where

$$C=2^{-1/2}\begin{bmatrix} 0 & 0 & b^{1/2} & b^{1/2} \\ b^{-1/2} & -b^{-1/2} & 0 & 0 \\ -b^{-1/2} & -b^{-1/2} & 0 & 0 \\ 0 & 0 & b^{1/2} & -b^{1/2} \end{bmatrix} \in \mathrm{Sp}(4, \boldsymbol{R}).$$

Then, according to (4.16), the sole eigenvalue $b$ of $-KA$ is the only single-mode energy and the most general single-mode stationary state is

$$\psi_b=\gamma(\overset{+}{a}_1\psi_0+\overset{+}{a}_2\psi_0) \qquad \text{with} \qquad \gamma\in C. \tag{4.20}$$

We can find another single-mode state $\psi_{\mathrm{D}}$, which is a principal vector for $H-\alpha$ and satisfies

$$(H-\alpha-b)\psi_D=\psi_b \tag{4.21$a$}$$

and

$$(H-\alpha-b)^2\psi_D=0. \tag{4.21$b$}$$

From (4.14) and (4.19), the vacuum energy $\alpha$ is equal to $b$. One solution for (4.21) is

$$\psi_D=\gamma b(\overset{+}{a}_1\psi_0-\overset{+}{a}_2\psi_0). \tag{4.22}$$

From the commutation relations (4.5e).

$$\langle\psi_b,\psi_b\rangle=0, \qquad \langle\psi_D,\psi_D\rangle=0, \qquad \langle\psi_D,\psi_b\rangle=2|\gamma|^2 b. \tag{4.23$a, b, c$}$$

Because of (4.21$a$) and (4.23$b$), $\psi_D$ is often called the 'dipole ghost'.

## 5. Interpretability of indefinite metric

We have seen in § 4 how a single-mode space with indefinite metric can be constructed rigorously by extending the mathematics of algebraic quantisation. The physical interpretation of the indefinite metric is a separate problem. This interpretation problem is no different from that which has prevailed over the last two decades, in the context of heuristic quantisation (Nagy 1966). It is generally agreed (e.g. Ascoli and Minardi 1958) that if the indefinite metric is to be given a probabilistic interpretation, the dynamically invariant single-mode space $\mathscr{H}^{(1)}$ must be decomposable into a direct sum $\mathscr{H}^{(1)}=\mathscr{H}_{\mathrm{p}}\oplus\mathscr{H}_{\mathrm{n}}$, with each element $\psi_{\mathrm{p}}$ of $\mathscr{H}_{\mathrm{p}}$ having a non-negative norm

$\langle\psi_p, \psi_p\rangle_1$. $\mathscr{H}_p$ represents the physical states of the system and $\mathscr{H}_n$ represents the non-physical states. Mathematically, there are many ways in which an indefinite inner product space can be decomposed in this way. For example, if $\mathscr{H}^{(1)}$ is two dimensional, $\langle\psi_1, \psi_1\rangle_1 = 1$, $\langle\psi_1, \psi_2\rangle_1 = 0$ and $\langle\psi_2, \psi_2\rangle_1 = -1$, $\mathscr{H}_p$ could be taken to be all scalar multiples of $\psi_1$, but could just as well be taken to be all scalar multiples of $\psi_1 + \frac{1}{2}\psi_2$. Since $\langle\psi_1 + \frac{1}{2}\psi_2, \psi_1 + \frac{1}{2}\psi_2\rangle_1 = \frac{3}{4}$, $\mathscr{H}_p$ would have positive metric after either choice. In practice, $\mathscr{H}_p$ is determined by the physics of the system which is represented. For example, in the Gupta–Bleuler covariant formulation of quantum electrodynamics, $\mathscr{H}_p$ includes the states representing transversely polarised photons, while $\mathscr{H}_n$ includes any state involving longitudinal or time-like photons. According to Ascoli and Minardi (1958), the minimal requirement for an indefinite inner product $\mathscr{H}^{(1)}$, with pseudo-unitary Hamiltonian dynamics, to have a probabilistic interpretation, is that if $\psi(0)$ belongs to $\mathscr{H}_p$, then for all $t$,

$$\psi(t) = e^{-Ht}\psi(0) = \psi_p(t) + \psi_n(t), \tag{5.1}$$

with

$$\psi_{p,n} \in \mathscr{H}_{p,n} \qquad \text{and} \qquad \langle\psi_p(t), \psi_n(t)\rangle_1 = \langle\psi_n(t), \psi_n(t)\rangle_1 = 0. \tag{5.2}$$

Equation (5.1) ensures that $\psi_n(t)$ in no way contributes to the norm of $\psi(t)$. However, it can easily be checked that these conditions cannot be met either for the case of complex frequencies or for the dipole ghost which we have just examined. In the former case, $\mathscr{H}^{(1)}$ is two dimensional, so that $\mathscr{H}_p$ must be one dimensional, spanned by a single state vector $\psi_1 = \xi\psi_E + \zeta\psi_{E^*}$, for some $\xi, \zeta \in C$. Assuming $\psi(0) = \psi_1$,

$$\psi(t) = \xi\, e^{-iEt}\psi_E + \zeta\, e^{-iE^*t}\psi_{E^*} \qquad \text{with } E = b + a\mathrm{i}.$$

From (5.1),

$$\psi_n(t) = \psi(t) - \mu\psi_1 \tag{5.3}$$

since $\mu\psi_1$, with $\mu \in C$, is an arbitrary element of $\mathscr{H}_p$. The condition $\langle\psi_n, \psi_n\rangle_1 = 0$ implies

$$0 = (1 + |\mu|^2)\langle\psi_1, \psi_1\rangle_1 - 2\,\mathrm{Re}(\mu\langle\psi(t), \psi_1\rangle_1). \tag{5.4}$$

From the condition $\langle\psi_n, \psi_p\rangle_1 = 0$, $\langle\psi(t), \psi_1\rangle_1 = \langle\mu\psi_1, \psi_1\rangle$, so that (5.4) becomes

$$(1 - |\mu|^2)\langle\psi_1, \psi_1\rangle_1 = 0. \tag{5.5}$$

Unless every physical state in $\mathscr{H}_p$ is to have zero norm, $\langle\psi_1, \psi_1\rangle_1 \neq 0$. Therefore, (5.5) implies

$$|\mu| = 1. \tag{5.6}$$

However, the requirement $\langle\psi_p, \psi_n\rangle_1 = 0$ implies $\langle\psi_1, \psi_n\rangle_1 = 0$, so that by (5.3),

$$\begin{aligned} &\langle\psi_1, \psi(t)\rangle_1 - \mu\langle\psi_1, \psi_1\rangle_1 = 0 \\ &\qquad \Rightarrow \mu = \langle\psi_1, \psi(t)\rangle_1/\langle\psi_1, \psi_1\rangle_1 \\ &\qquad\qquad = \mathrm{Re}(2\beta^*\gamma\zeta^*\xi\, e^{-ibt}\, e^{at} + 2\beta\gamma^*\zeta\xi^*\, e^{-ibt}\, e^{-at})/\langle\psi_1, \psi_1\rangle_1, \end{aligned} \tag{5.7}$$

using (4.18).

This exponential increase in the amplitude of $\mu$ is impossible to reconcile with $|\mu| = 1$. This shows that in the case of simple complex frequencies, the usual requirements of physical interpretability cannot be met.

In the dipole ghost situation of (4.21), $\mathscr{H}^{(1)}$ is again two dimensional, so that $\mathscr{H}_p$ must consist of scalar multiples of a single state $\psi_1 = \xi\psi_b + \zeta\psi_D$. Recalling that in (4.21), $\alpha = b$, we have

$$H\psi_b = 2b\psi_b, \qquad e^{-iHt}\psi_b = e^{-2ibt}\psi_b, \tag{5.8a}$$

$$H\psi_D = \psi_b + 2b\psi_D, \qquad H^2\psi_D = 4b\psi_b + (2b)^2\psi_D, \tag{5.8b}$$

$$H^n\psi_D = n(2b)^{n-1}\psi_b + (2b)^n\psi_D, \qquad e^{-iHt}\psi_D = e^{-2ibt}\psi_D - it\, e^{-2ibt}\psi_b.$$

Assuming $\psi(0) = \psi_1$ and using (5.8),

$$\psi(t) = \xi\, e^{-2ibt}\psi_b + \zeta\, e^{-2ibt}\psi_D - it\zeta\, e^{-2ibt}\psi_b = \mu\psi_1 + \psi_n(t),$$

since every element of $\mathscr{H}_p$ has the form $\mu\psi_1$ with $\mu \in C$. Therefore

$$\psi_n(t) = \psi(t) - \mu\psi_1 = (e^{-2ibt} - \mu)\psi_1 - it\zeta\, e^{-2ibt}\psi_b. \tag{5.9}$$

Hence

$$\begin{aligned}\langle\psi_n(t), \psi_p(t)\rangle_1 &= \mu(e^{2ibt} - \mu^*)\langle\psi_1, \psi_1\rangle_1 + \mu\, it\zeta^*\, e^{2ibt}\langle\psi_b, \psi_1\rangle_1\\ &= 2|\gamma|^2 b(\xi^*\zeta + \xi\zeta^*) + 2\mu\, it|\zeta|^2|\gamma|^2 b\, e^{2ibt},\end{aligned}$$

by (4.23).

From (4.23), the requirement $\langle\psi_n, \psi_p\rangle = 0$ implies $\zeta = 0$. Therefore, an arbitrary physical state $\mu\psi_1$ has the form $\mu\xi\psi_b$, which has vanishing norm, by (4.23*a*). Therefore, this norm could not be interpretable as a probability. In Heisenberg's treatment of the Lee model (Heisenberg 1957), the dipole ghost occurs in two degrees of freedom of an interacting field. The Hamiltonian can be unambiguously partitioned into a free term and an interaction term. This leads to the definition of an $S$-operator which maps incoming physical states onto outgoing physical states of the same norm. Thus, in the words of Heisenberg, although a physical interpretation of the local behaviour in terms of probabilities cannot be given, it is conceivable that such a model might be adequate for a scattering experiment.

In this paper, we have exposed the following disadvantages in allowing the indefinite metric to enter algebraic quantisation.

(a) When the requirement of positive metric is discarded, the unitarising complex structure, when it exists, is no longer uniquely determined by the classical dynamics and neither is the signature of the metric.

(b) Any advantage of allowing the metric to be indefinite is limited by the fact that certain simple classical systems, such as the free particle in one dimension, still cannot be (pseudo)-unitarised.

(c) When an unstable system can be pseudo-unitarised, the local behaviour cannot be interpreted in terms of probabilities.

To this list, we may add another problem unearthed by Araki (1982).

(d) In order to analyse a Hamiltonian $H$ on Fock space with indefinite metric, we close $H$ in the topology determined by some chosen constructed positive metric. However, the spectrum of the closure $\bar{H}$ depends, in a dramatic way, on the choice of topology.

Despite the above list of difficulties, the physical literature still kindles some hope for eventual success in interpretation. One area in which the indefinite metric has been encountered is in non-local field theory. For example, consider a scalar field

$\phi(x)$ ($x$ belonging to Minkowski space), satisfying the equation

$$\Box^2\phi(x)=0 \qquad (\Box \text{ is the d'Alembertian wave operator } \Box=\partial_t^2-\Delta, \text{ with } \Delta \text{ the Laplacian}). \tag{5.10}$$

This is the equation satisfied by each component of the four-potential for the free electromagnetic field in the Landau gauge (Carey and Hurst 1978). To see that (5.10) is grossly non-local, note that it is equivalent to

$$\Box\phi(x)=j(x), \tag{5.11}$$

where the non-localised source term $j(x)$ is an arbitrary solution of the Klein–Gordon equation. The equation (5.11) is macroscopically acausal, since its solutions include not only the Klein–Gordon field, but also the harmonic field (d'Emilio and Mintchev 1979), which results in a Wightman two-point function which is proportional to $(x-y)^2=(x-y)^\mu(x-y)_\mu$, resulting in non-vanishing commutators $[\phi(x), \phi(y)]$, with $x-y$ space-like.

Serious interest has been shown in equation (5.10) as a model of the sub-hadronic gluon field (e.g. d'Emilio and Mintchev 1979, Narnhofer and Thirring 1978). A fundamental Green function for the operator $-\Delta^2$ is $r=(\boldsymbol{x}\cdot\boldsymbol{x})^{1/2}$

$$-(8\pi)^{-1}\Delta^2 r=\delta(x). \tag{5.12}$$

Therefore, (5.10) is a very simple example of a system with confining static potential in the presence of a point source. Following the programme of heuristic quantisation for the system (5.10), Narnhofer and Thirring (1978) found the formal Hamiltonian to be

$$H=\sum_{\boldsymbol{k}} H_{\boldsymbol{k}},$$

with

$$H_{\boldsymbol{k}}=\tfrac{1}{2}\boldsymbol{\alpha}^\dagger(\boldsymbol{k})\mathscr{D}_1(\boldsymbol{k})\boldsymbol{\alpha}(\boldsymbol{k}), \qquad \boldsymbol{\alpha}^\dagger(\boldsymbol{k})=(a_1(\boldsymbol{k})^\dagger, a_2(\boldsymbol{k})^\dagger, a_1(\boldsymbol{k}), a_2(\boldsymbol{k})) \tag{5.13}$$

with

$$\mathscr{D}_1(\boldsymbol{k})=\begin{bmatrix} (4k_0)^{-1} & k_0-\mathrm{i}(4k_0)^{-1} & (4k_0)^{-1} & \mathrm{i}(4k_0)^{-1} \\ k_0+\mathrm{i}(4k_0)^{-1} & (4k_0)^{-1} & \mathrm{i}(4k_0)^{-1} & -(4k_0)^{-1} \\ (4k_0)^{-1} & -\mathrm{i}(4k_0)^{-1} & (4k_0)^{-1} & k_0+\mathrm{i}(4k_0)^{-1} \\ -\mathrm{i}(4k_0)^{-1} & -(4k_0)^{-1} & k_0-\mathrm{i}(4k_0)^{-1} & (4k_0)^{-1} \end{bmatrix},$$

with $k_0=|\boldsymbol{k}|$.

In (5.13), the operators $a_j(\boldsymbol{k})$ are ordinary boson annihilation operators:

$$[a_j(\boldsymbol{k}), a_l(\boldsymbol{k}')^\dagger]=\delta_{jl}\delta(\boldsymbol{k}-\boldsymbol{k}'). \tag{5.14}$$

Therefore, according to the scheme of Broadbridge (1979), the corresponding classical Hamiltonian is

$$H=\sum_{\boldsymbol{k}}\tfrac{1}{2}z(\boldsymbol{k})^{\mathrm{T}}\hat{H}(\boldsymbol{k})z(\boldsymbol{k}), \qquad \hat{H}(\boldsymbol{k})=\begin{bmatrix} (2k_0)^{-1} & k_0 & 0 & (2k_0)^{-1} \\ k_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_0 \\ (2k_0)^{-1} & 0 & k_0 & (2k_0)^{-1} \end{bmatrix}. \tag{5.15}$$

This Hamiltonian can be classified according to the scheme of § 2. With $\hat{H} = \hat{H}(\boldsymbol{k})$, the elementary divisors of $iG\hat{H} - sI$ are $(s \pm k_0)^2$. Therefore, $\hat{H}$ must belong to one of the two canonical orbits with canonical form $\hat{K}_6^{(4)}(\rho)$, with $\rho = \pm 1$. To determine $\rho$, we must find the signature of $\hat{H}_N$, given that $-G\hat{H}_N$ is the nilpotent part in the Jordan decomposition of $-G\hat{H}$. The matrix $\hat{H}_N$ is given by

$$\hat{H}_N = \mathrm{diag}[(2k_0)^{-1}, 0, 0, (2k_0)^{-1}], \tag{5.16}$$

which is verified simply by checking that $-G\hat{H}_N$ is nilpotent and that $-G\hat{H}_N$ commutes with $-G\hat{H}$. Therefore, $\hat{H}_N$ is positive semi-definite and $\rho = +1$. The determination of the canonical orbit of $\hat{H}_{\boldsymbol{k}}$ immediately leads to further information on the quantum mechanical system. The quadratic elementary divisors and real frequencies $k_0$ lead to linear instability in the evolution of field operators $\phi(\boldsymbol{x}, t)$. Since the elementary divisors are not linear, algebraic unitarisation of the classical dynamics is not possible. However, there exists a complex structure which enables the classical dynamics to be pseudo-unitary, since all frequencies are real and non-vanishing. As pointed out by Narnhofer and Thirring, the one parameter group of Bogoliubov transformations generated by (5.13) is not unitarily implementable on Fock space. Equivalently, the formal Hamiltonian (5.13) cannot be closed and then extended to a self-adjoint operator. Despite all these difficulties, Thirring and Narnhofer have provided a novel suggestion as to how consistency might be regained. Namely, if (5.10) is to be a gluon field, then other interaction terms must be introduced, since the gluon field is not isolated. Just as in the case of the external field problem, in which a healthy free field develops instability when an interaction is introduced, a pathological free field may be stabilised by introducing an interaction. An initial exploratory model, in which the extra field is represented by a harmonic oscillator, has shown some success. Since a fully interacting Hamiltonian $H$ may be decomposed into a 'free' term $H_0$ of the form (5.13) and an interaction term, the $S$-operator may be defined from a counterpart of $e^{iHt}\, e^{-iH_0 t}$ and this can be unitary, even though the original Fock space has indefinite metric. The process of extracting a unitary $S$-matrix out of a pseudo-unitary dynamical system has recently been further developed by Demuth (1981). The sub-hadronic realm may yet prove to require strange mathematics in its description.

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